Validated Numerical Newton–Puiseux Algorithm

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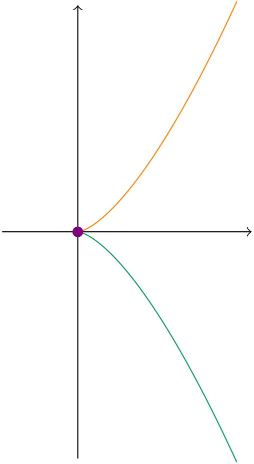


$F(X, Y) = Y^2 - X^3$

Puiseux series of F: $X^{\frac{3}{2}}$ and $-X^{\frac{3}{2}}$

$F \in \mathbb{Q}[X][Y]$ monic, 0 singular point

Puiseux series of F:

$$Y(X) = \sum_{k \geq 0} a_k X^{\frac{k}{e}}$$

for $e \in \mathbb{N}$ and $a_k \in \overline{\mathbb{Q}}$

Newton–Puiseux algorithm

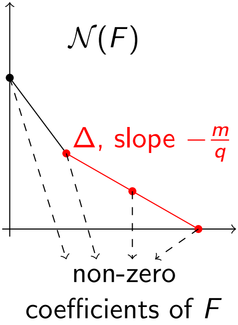
$$Y(X) = \alpha X^{\frac{m}{q}} + \ldots \text{ s.t. } F(X, Y(X)) = 0$$

1. Each edge Δ of the Newton polygon $\mathcal{N}(F)$
   $\implies$ provides an exponent $\frac{m}{q}$
2. For each root ξ_i of the characteristic polynomial

   $$\phi_\Delta = \prod_{i=0}^{s} (Z - \xi_i)^{\mu_i}$$

   $\implies$ provides $\alpha = \xi_i^{\frac{1}{q}}$
3. Higher order terms are recovered with a recursive call after the change of variable:

$$\left\{ \begin{array}{lcc} X & \leftarrow & X^q \\ Y & \leftarrow & X^m(\xi_i^{\frac{1}{q}} + Y) \end{array} \right.$$



Symbolic–numeric strategy (Poteaux Rybowicz, 2008)

Coefficient growth + algebraic numbers

$\Longrightarrow$ Walsh 2000: $\tilde{\mathcal{O}}(d^{36})$ bits operations (d total degree of F)

- Newton–Puiseux mod $p \Longrightarrow$ structure of Puiseux series
  - Newton polygon
  - Multiplicity structures
- Numerical computations of coefficients of the Puiseux series following the structure

Today: use the framework of validated numerics

$$[1.33 \pm 0.01]X^{\frac{1}{3}} + [7.81 \pm 0.03]X^{\frac{5}{8}} + [9.87 \pm 0.03]X^{\frac{11}{8}} + \dots$$

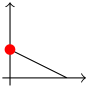
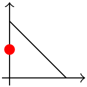
Validated Numerical Part (My PhD)

Problem: no isomorphism between $\mathbb{F}_p$ and $\mathbb{C}$

Over $\mathbb{Q}$	Over $\mathbb{F}_{29}$	?	Over $\mathbb{C}$
$\frac{4023}{2114}X^{\frac{1}{2}} + \frac{78}{61}X^{\frac{3}{4}} + \dots$	$22X^{\frac{1}{2}} + \dots$		$[1.9 \pm 0.01]X^{\frac{1}{2}} + \dots$
$\frac{6038}{3027}X^{\frac{1}{2}} + \frac{3}{14}X^{\frac{5}{4}} + \dots$	$19X^{\frac{1}{2}} + \dots$		$[2 \pm 0.01]X^{\frac{1}{2}} + \dots$

Solution:

- Polygon filter $\Longrightarrow$

- Multiplicity filter $\Longrightarrow$

$$\phi_1(Z) = (Z - \alpha)^3$$
$$\phi_2(Z) = (Z - \beta_1)^1 (Z - \beta_2)^2$$

Validation of coefficients:

- A posteriori validation method (Krawczyk, ...)
- Ball arithmetic

Contributions & Future works

Contributions:

- Submitted to ISSAC 2026: https://hal.science/hal-05510315
  - Correctness and termination of the algorithm
  - Output intervals can be made as tight as desired
  - Arithmetic complexity in $\tilde{\mathcal{O}}(d^4)$ ball operations
- Implemented in Julia (based on Nemo): https://gitlab.univ-lille.fr/adrien.poteaux/numpuiseux.jl

Future works:

- A posteriori validation version of Puiseux $\tilde{\mathcal{O}}(d^4)$
- Validated Numerical OM algorithm (Puiseux $\tilde{\mathcal{O}}(d^3)$)
- Binary complexity of the algorithm
- Applications to other fields s.a. robotics or physics