Validated Numerical Newton–Puiseux Algorithm

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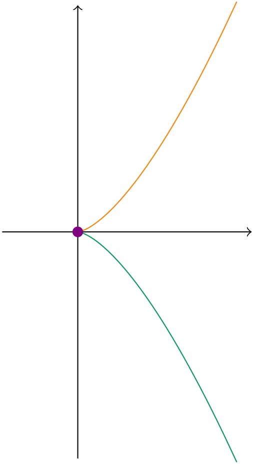
Puiseux series are a tool to analyze plane curves at singularities



Theorem (Puiseux, 1850)

$F \in \mathbb{Q}[X][Y]$ monic,
Puiseux series of F above 0:

$$Y(X) = \sum_{k \geq 0} a_k X^{\frac{k}{e}}$$

with $e \in \mathbb{N}$ and $a_k \in \overline{\mathbb{Q}}$

$F(X, Y) = Y^2 - X^3$
$(0, 0)$ is singular $\Longrightarrow$ We have Puiseux series

Puiseux series of F:
$Y(X) = X^{\frac{3}{2}}$ and $Y(X) = -X^{\frac{3}{2}}$

Motivations

Puiseux series

Avoiding generic assumptions

e.g. [27] Prébet

$\mathbb{R}$

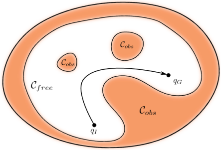
Robotics:

Connectivity & Topology
Piano mover's problem

$\mathbb{C}$

Integrals near singularities

[7] Deconinck & Paterson



Effective Abel-Jacobi theorem

Computer algebra:

Integration of algebraic functions
. . .

Physics:

KdV & KP equations

Validated Numerical Newton–Puiseux algorithm

Let d the total degree of $F \in \mathbb{Q}[X][Y]$

- Arithmetic complexity:
  - [9] Duval $\rightarrow \mathcal{O}^{\sim}(d^8)$ (Rational Puiseux Expansions)
  - [22], [23], [25] Poteaux & al. $\rightarrow \mathcal{O}^{\sim}(d^5), \mathcal{O}^{\sim}(d^4), \mathcal{O}^{\sim}(d^3)$
- Bit complexity over $\mathbb{Q}$:
  Coefficient growth + algebraic numbers
  $\implies$ [37] Walsh: $\mathcal{O}^{\sim}(d^{36})$ bits operations
- Modular–numeric: [23] Poteaux & Rybowicz
  - Newton–Puiseux mod $p \implies$ structure of Puiseux series
  - Numerical computations of coefficients following the structure
    $\rightarrow$ No certification of coefficients

This paper: use the framework of validated numerics
(Interval arithmetic + A posteriori validation method)

Other related work: [16], [30], [26]
Differential equation, Extended Hensel lemma, OM algorithm, ...

Newton–Puiseux algorithm: main tools

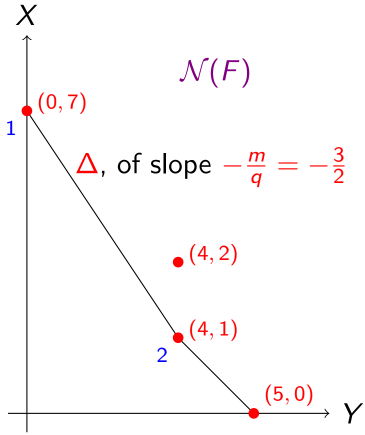
$$F = \sum a_{ij} X^{j} Y^{i}$$
$$= Y^{5} + X^{2} Y^{4} + 2X^{1} Y^{4} + 1X^{7}$$

$\bullet$ = Non-zero coeffs of F

Newton polygon $\mathcal{N}(F)$
= Lower convex hull of $\bullet$

For each $\Delta \in \mathcal{N}(F)$,

$$\phi_\Delta = \sum_{(i,j)\in\Delta} a_{ij} Z^{\frac{i-i_0}{q}},\ i_0 = \min_{(i,j)\in\Delta} i$$
$$= 2Z^{\frac{4}{2}} + 1$$
$$= \text{characteristic polynomial}$$



Newton–Puiseux algorithm

$$Y(X) = \alpha X^{\frac{m}{q}} + \ldots \text{ s.t. } F(X, Y(X)) = 0$$

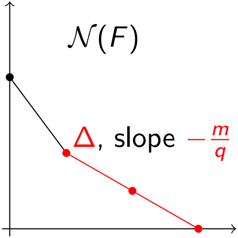
1. Each edge Δ of the Newton polygon $\mathcal{N}(F)$
   $\implies$ provides an exponent $\frac{m}{q}$
2. For each root ξ_i of the characteristic polynomial

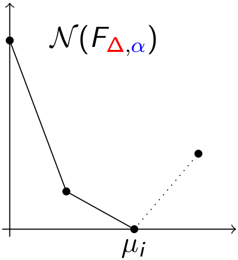
$$\phi_\Delta = \prod_{i=0}^{s} (Z - \xi_i)^{\mu_i}$$

   $\implies$ provides $\alpha = \xi_i^{\frac{1}{q}}$
3. Higher order terms are recovered with a recursive call after the change of variable:

$$F_{\Delta,\alpha} = \begin{cases} X & \leftarrow & X^q \\ Y & \leftarrow & X^m(\alpha + Y) \end{cases}$$

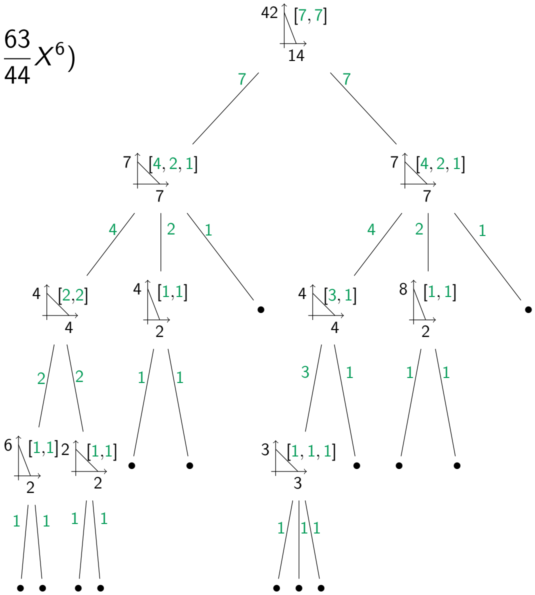
Stops when $\mu_i = 1$





$$F = \left(Y - \frac{59}{52}X^3 - \frac{25}{8}X^4 - \frac{54}{97}X^5 - \frac{63}{44}X^6\right)$$
$$\times \left(Y - \frac{16}{15}X^3 + \frac{38}{21}X^4\right)$$
$$\times \ldots \qquad \text{[14 Puiseux series]}$$

1 Node $= 1$ call



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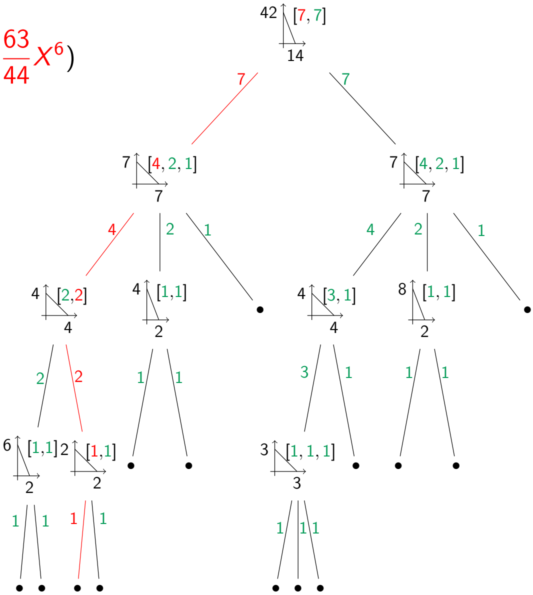

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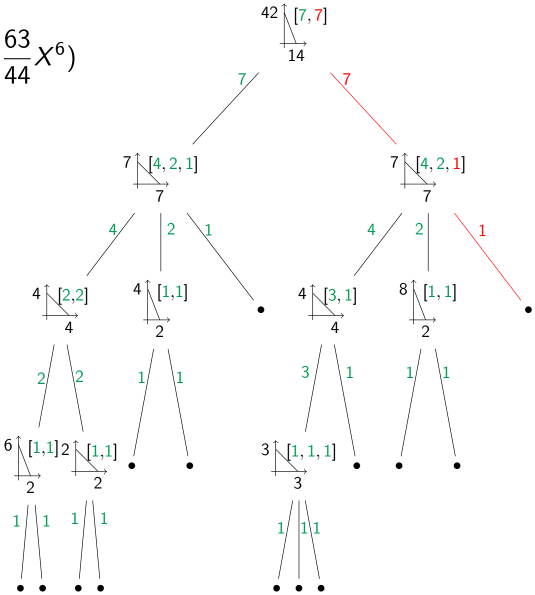

Polygon trees – Subsection 3.3

$$F = \left(Y - \frac{59}{52}X^3 - \frac{25}{8}X^4 - \frac{54}{97}X^5 - \frac{63}{44}X^6\right)$$
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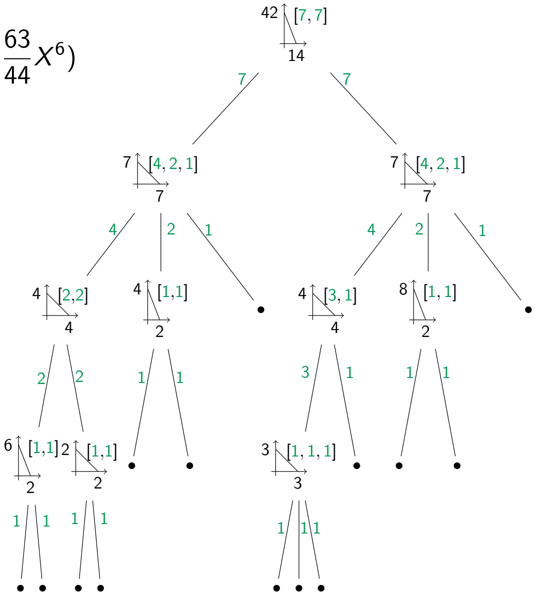
Criterion $\Rightarrow$ 29 is a good prime



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$$\times \ldots \qquad [14 \text{ Puiseux series}]$$

$\frac{59}{52}$ and $\frac{16}{15}$ are of multiplicity 7

Which subtree $\leftrightarrow$ to which root ?

We can't decide, we only have:

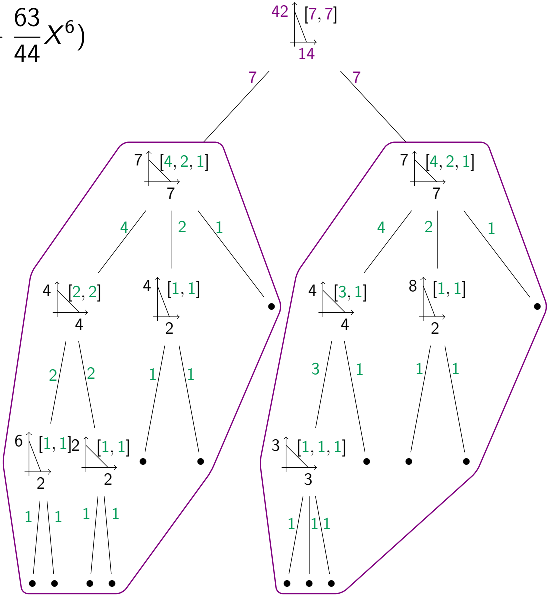
– enclosures of roots:

$[1.13 \pm 0.01]$ and $[1.06 \pm 0.01]$

– values of roots mod 29:

24 and 3

Each ball contains infinitely many rationals equals to 24 and 3 mod 29



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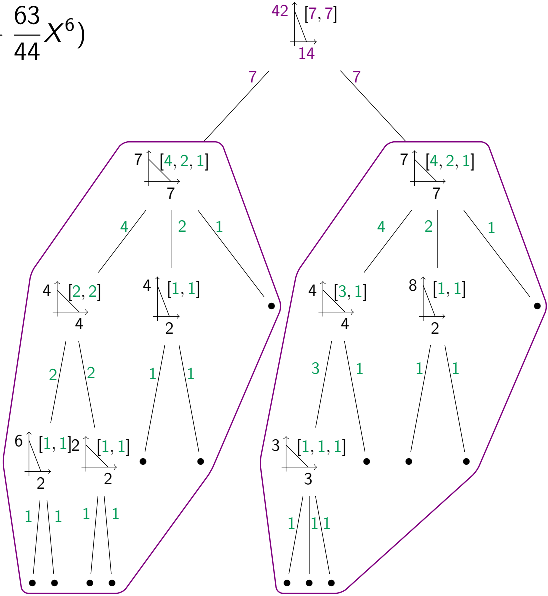
Criterion ⇒ 29 is a good prime

Several roots of same multiplicity

⇒ We work with collections



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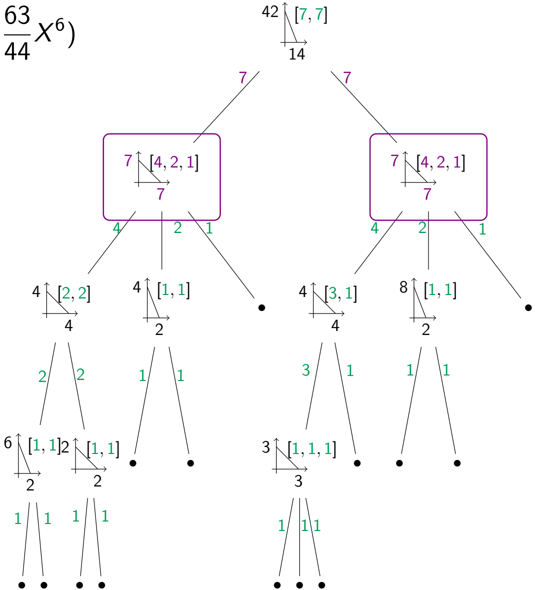
Criterion ⇒ 29 is a good prime

Several roots of same multiplicity

⇒ We work with collections

⇒ Choose the correct candidate

Solutions:



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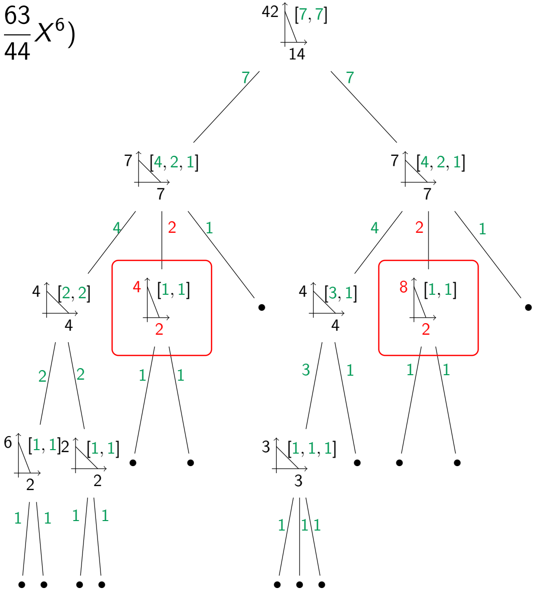
Solutions:

- Polygon Filter



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Criterion $\Rightarrow$ 29 is a good prime

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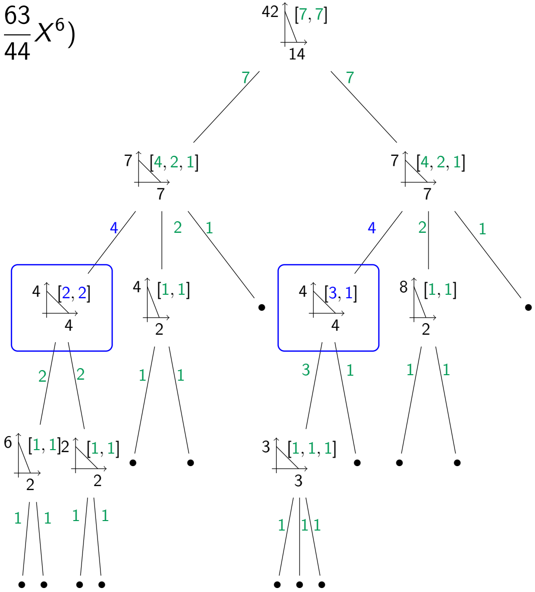
$\Rightarrow$ We work with collections

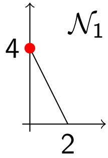
$\Rightarrow$ Choose the correct candidate

Solutions:

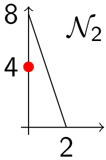
- Polygon Filter
- Multiplicity Filter



Polygon filter



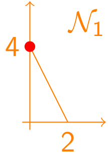
$$\ldots + 2X^4 + \ldots$$



$$\ldots + 0X^4 + \ldots$$

$$\mathbf{P_1} = \ldots + [1 \pm 4]X^4 + \ldots \qquad \mathbf{P_2} = \ldots + [-2 \pm 3]X^4 + \ldots$$

Not Enough precision to do the polygon filter !
We restart the algorithm with better precision



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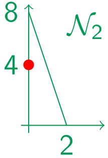

$$\ldots + 0X^4 + \ldots$$

$$\mathbf{P_1} = \ldots + [0.3 \pm 1]X^4 + \ldots \qquad [0.3 \pm 1] \ni 0$$

$$\mathbf{P_2} = \ldots + [2 \pm 1]X^4 + \ldots \qquad [2 \pm 1] \not\ni 0$$

Multiplicity filter

[1, 1, 1]	[2, 1]
$\phi_1 = (Z - \frac{21}{53})^1 (Z - \frac{90}{11})^1 (Z - \frac{43}{8})^1$	$\phi_2 = (Z - \frac{39}{15})^2 (Z - \frac{54}{11})^1$

$$\mathbf{\Phi_1} = Z^3 - [12.9 \pm 0.7]Z^2 + [48.4 \pm 0.6]Z - [17.5 \pm 0.1]$$

$$\mathbf{\Phi_2} = Z^3 - [10.5 \pm 1.1]Z^2 + [30.2 \pm 1.5]Z - [32.1 \pm 0.7]$$

Data: A list of **polynomials** and a list of multiplicity structures

Goal: Match each **polynomial** with its multiplicity structure

Multiplicity filter

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Data: A list of **polynomials** and a list of multiplicity structures
Goal: Match each **polynomial** with its multiplicity structure

1. Associate number of distinct roots
2. Extract roots
3. Find multiplicities

Associate number of distinct roots - Some facts

Let $0 \leq k \leq n$, and let $S(\phi, k)$ be the system with unknowns v, w,

$$\underbrace{v}_{Z^k+\dots}\phi' - \underbrace{w}_{nZ^{k-1}+\dots}\phi = 0$$

- ϕ has less than k roots $\Rightarrow S(\phi, k)$ has at least 2 solutions

- ϕ has k roots $\Rightarrow S(\phi, k)$ has a unique solution (v, w) s.t.
  $$\{\xi_i\} = \underset{\text{multiplicity } 1}{\underset{\downarrow}{\text{Roots of } v}} = \underset{\text{multiplicity } \mu_i}{\underset{\downarrow}{\text{Roots of } \phi}} \quad \text{and } \frac{w(\xi_i)}{v'(\xi_i)} = \mathrm{Mult}(\xi_i, \phi)$$

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Also, we have a validated numerical procedure $\mathbf{\Gamma}(\mathbf{\Phi}, k)$, such that:

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Also, we have a validated numerical procedure $\mathbf{\Gamma}(\mathbf{\Phi}, k)$, such that:

- ϕ has less than k roots $\Rightarrow S(\phi, k)$ has at least 2 solutions

  $\mathbf{\Phi}$ contains such a $\phi \Rightarrow \mathbf{\Gamma}(\mathbf{\Phi}, k)$ necessarily returns an error

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  All polynomials in $\mathbf{\Phi}$ have at least k roots $\Rightarrow$
  $\mathbf{\Gamma}(\mathbf{\Phi}, k)$ returns a correct enclosure of the solutions of $S(\phi, k)$ for any $\phi \in \mathbf{\Phi}$ with exactly k roots

Associate number of distinct roots

$[1, 1, 1]$	$[2, 1]$
$\phi_1 = (Z - \frac{21}{53})^1 (Z - \frac{90}{11})^1 (Z - \frac{43}{8})^1$	$\phi_2 = (Z - \frac{39}{15})^2 (Z - \frac{54}{11})^1$

$$\mathbf{\Phi_1} = Z^3 - [12.9 \pm 0.7]Z^2 + [48.4 \pm 0.6]Z - [17.5 \pm 0.1]$$

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Decision criterion:

$\mathbf{\Phi_i}$ the polynomial associated with $\phi_1 \Rightarrow \mathbf{\Gamma}(\mathbf{\Phi_i}, 3)$ succeeds with enough precision

$\mathbf{\Phi_j}$ the polynomial associated with $\phi_2 \Rightarrow \mathbf{\Gamma}(\mathbf{\Phi_j}, 3)$ always return an error

Associate number of distinct roots

$$[1, 1, 1]$$

$$\phi_1 = (Z - \frac{21}{53})^1 (Z - \frac{90}{11})^1 (Z - \frac{43}{8})^1$$

$$[2, 1]$$

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$$\uparrow \qquad \uparrow$$

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Main Result

ValidatedPuiseux($\mathcal{H}, \mathcal{L}$)

In:

- $\mathcal{H}$ a list of interval polynomials
- $\mathcal{L}$ a list of polygon trees

Out:

Interval Puiseux series

1. Polygon filter
2. Multiplicity filter
3. Change of variable
4. Recursive call

Proposition 6.1

Let $F \in \mathbb{Q}[X][Y]$ and $\mathcal{T}(F)$ his polygon tree.

(i) If ValidatedPuiseux($[F], [\mathcal{T}(F)]$) does not fail, then it computes a collection $\mathcal{R}$ of ball Puiseux series and all Puiseux series of F are contained in exactly one ball Puiseux series of $\mathcal{R}$.

(ii) ValidatedPuiseux($[F], [\mathcal{T}(F)]$) is guaranteed to succeed for a sufficiently high precision. Output intervals can be made as tight as desired.

(iii) Arithmetic complexity in $\mathcal{O}\tilde{}(d^4)$ FP/ball operations.

Conclusion & Future works

Contributions:

- Correctness and termination of the algorithm
- Output intervals can be made as tight as desired
- Arithmetic complexity in $\tilde{\mathcal{O}}(d^4)$ interval operations

- Implemented in Julia (based on Nemo): https://gitlab.univ-lille.fr/adrien.poteaux/numpuiseux.jl

Future work:

- A posteriori validation version of Puiseux $\tilde{\mathcal{O}}(d^4)$
- Validated Numerical OM algorithm (Puiseux $\tilde{\mathcal{O}}(d^3)$)
- Bit complexity of the algorithm
- Applications to other fields such as robotics or physics

$\mathbf{x}$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{b}$, with $\mathbf{A}$ rectangular with interval coefficients

1. Floating point approx. $\tilde{x}$
   $\Longrightarrow$ Least-square method
2. Construction of an error bound on $x \in E_r = [\tilde{x} \pm r], r_i > 0$
   x is a fixed point of $T(x) = x - C(Ax - b), C \approx A^*(AA^*)^{-1}$

Theorem (Krawczyk, Banach fixed point)

If we find E_r s.t.

$$T(E_r) \subset \mathring{E}_r$$

then E_r has a unique fixed point.

Future works: robotics

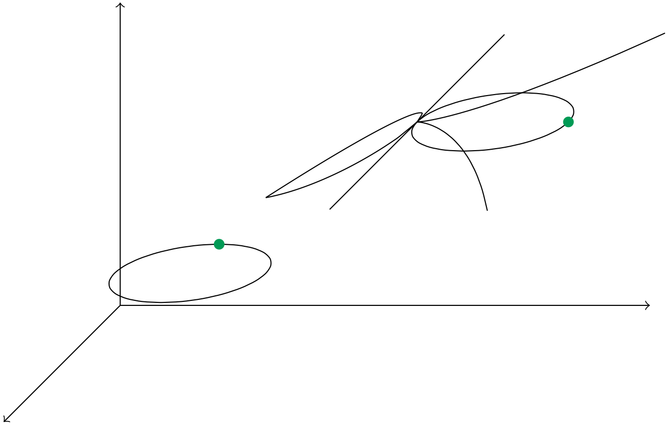
Problem of connectivity queries on semi-algebraic sets:

Given 2 points on a semi-algebraic set, determine whether there exists a path between them.

1. Due to the work of Canny, roadmaps reduce this problem to connectivity queries on (non-necessarily plane) curves.
2. Projection onto a plane reduces this problem to analyze a plane curve.

Plane curves have singularities, and Puiseux series provide a way to avoid generic projection.
Expectation: Same complexity without generic projection.

Future works: robotics



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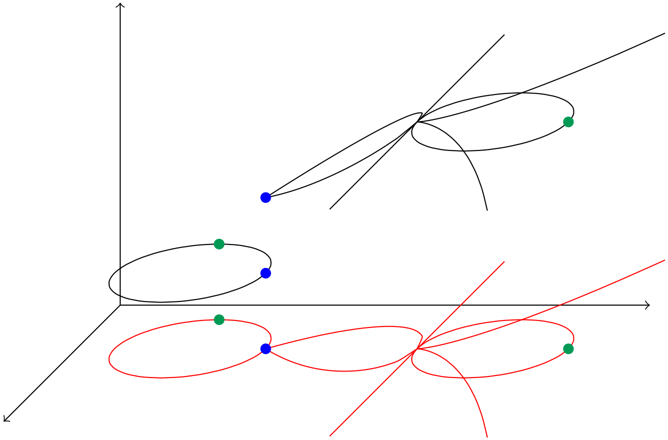

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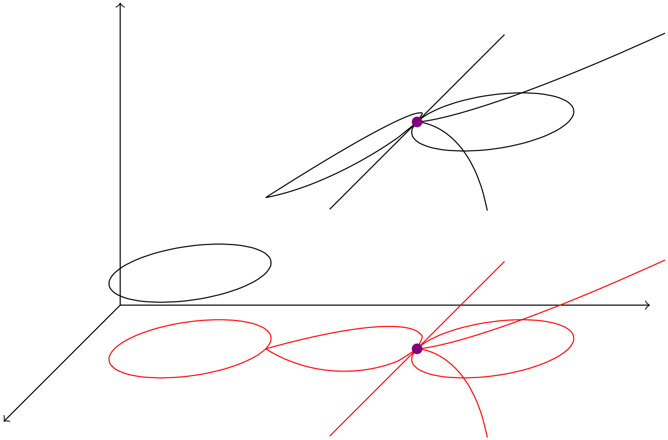

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